

Thinking task: four digits

2.16 · Mixed Differentiation · Tangent and Normal Lines

Use each digit **1, 2, 3, 4** exactly once. Make two two-digit numbers.

What is the **smallest positive difference**? Justify.

$$31 - 24 = 7$$

Tens contribute at least 10.

Ones can reduce that gap by at most 3.

$$10 - 3 = 7$$

Do-Now

The functions are differentiable wherever needed.

$$h(x) = f(x)g(x) + f(g(x))$$

Input	f value	f slope	g value	g slope
2	1	-3	4	2
4	5	$\frac{1}{2}$	—	—

Find $h(2)$, $h'(2)$, and the **tangent line at input 2**.

Choose each rule from the structure

$$h(2) = 1 \cdot 4 + 5 = 9$$

$$h'(2) = f'(2)g(2) + f(2)g'(2) + f'(g(2))g'(2)$$

$$h'(2) = (-3)(4) + (1)(2) + \frac{1}{2}(2) = -9$$

$$y - 9 = -9(x - 2)$$

The normal uses the same point

A normal line passes through the point and is **perpendicular to the tangent line**.

$$P = (a, f(a)), \quad m = f'(a) \neq 0$$

$$\text{Tangent: } y - f(a) = m(x - a)$$

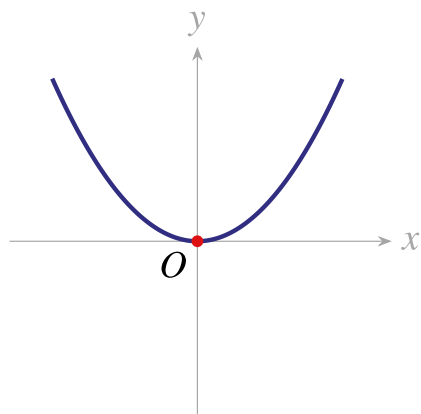
What slope must the normal have?

$$\text{Normal: } y - f(a) = -\frac{1}{m}(x - a)$$

$$m = -9 \quad \Rightarrow \quad m_N = \frac{1}{9}$$

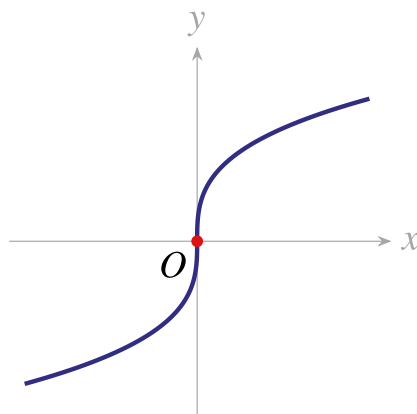
Three kinds of behavior at the origin

$$y = x^2$$



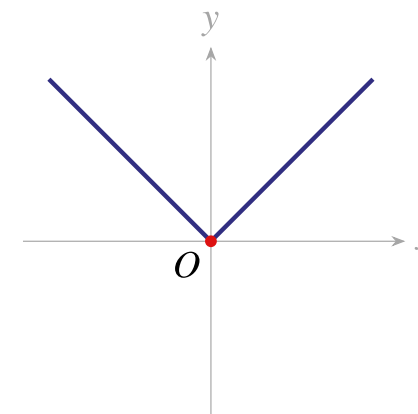
Parabola y equals x squared at the origin.

$$x = y^3$$



Smooth curve x equals y cubed at the origin.

$$y = |x|$$

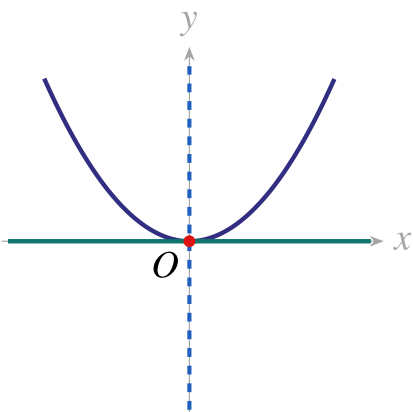


Absolute value curve with a corner at the origin.

At the origin, identify the tangent and normal, or explain why there is **no unique line**.

Undefined does not always mean vertical

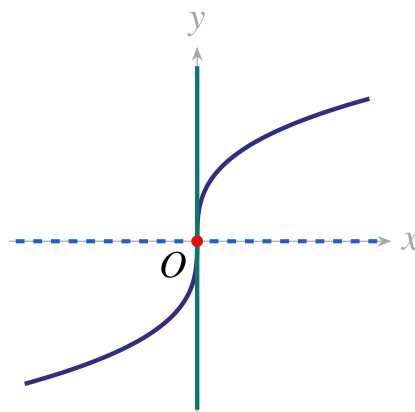
$$y = x^2$$



Parabola with a horizontal tangent and vertical normal.

$$\text{T: } y = 0 \quad \text{N: } x = 0$$

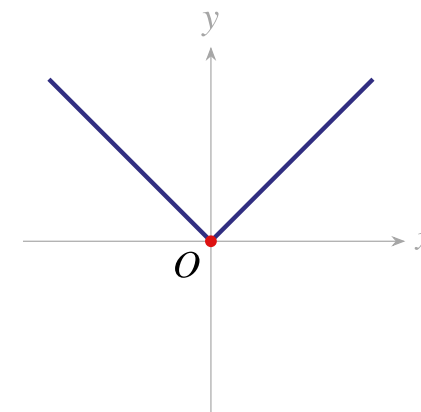
$$x = y^3$$



Curve x equals y cubed with a vertical tangent and horizontal normal.

$$\text{T: } x = 0 \quad \text{N: } y = 0$$

$$y = |x|$$



Absolute value curve; no unique tangent or normal at the corner.

Unequal one-sided slopes.
No single tangent or normal.

Solid teal: tangent · Dashed blue: normal

One function, two derivatives, two lines

$$q(x) = x^2 \ln x, \quad x > 0$$

Find the **first and second derivatives**.

At input 1, find the **tangent and normal lines**.

Keep the value and the slopes distinct

$$q'(x) = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x$$

$$q''(x) = 2 \ln x + 2x \frac{1}{x} + 1 = 2 \ln x + 3$$

$$q(1) = 0, \quad q'(1) = 1, \quad q''(1) = 3$$

$$\text{T: } y = x - 1 \quad \text{N: } y = 1 - x$$

C1 · Read the geometry backward

At every point with positive input, the tangent meets the y -axis **exactly one unit below the point**.

$$P = (a, f(a)), \quad a > 0$$

For what value of a are the y -intercepts of the tangent and normal **10 units apart**?

Translate the tangent condition

$$y - f(a) = f'(a)(x - a)$$

$$x = 0 \quad \Rightarrow \quad b_T = f(a) - af'(a)$$

$$f(a) - af'(a) = f(a) - 1$$

$$af'(a) = 1 \quad \Rightarrow \quad f'(a) = \frac{1}{a}$$

Now compare the two intercepts

$$m_N = -\frac{1}{f'(a)} = -a$$

$$y - f(a) = -a(x - a) \quad \Rightarrow \quad b_N = f(a) + a^2$$

$$b_N - b_T = (f(a) + a^2) - (f(a) - 1) = a^2 + 1$$

$$a^2 + 1 = 10, \quad a > 0 \quad \Rightarrow \quad a = 3$$

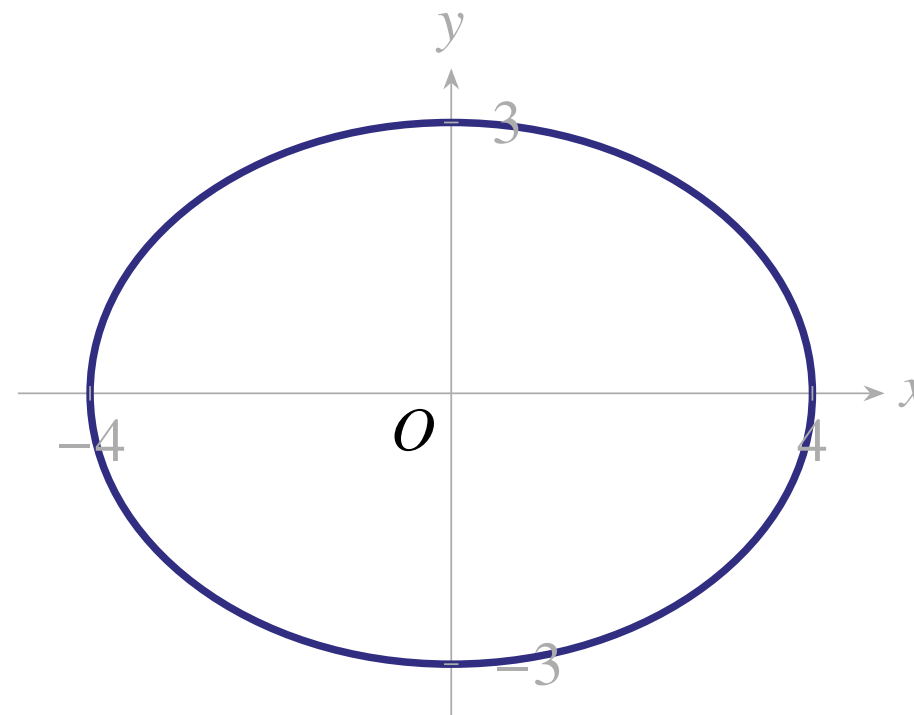
C2 · Does centered mean normal?

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

“Every normal passes through the origin because the curve is centered there.”

Decide whether the claim is true.

Find **every point** where it is true.



Ellipse with semiaxes 4 and 3 centered at the origin, without solution lines.

First consider a point off both axes

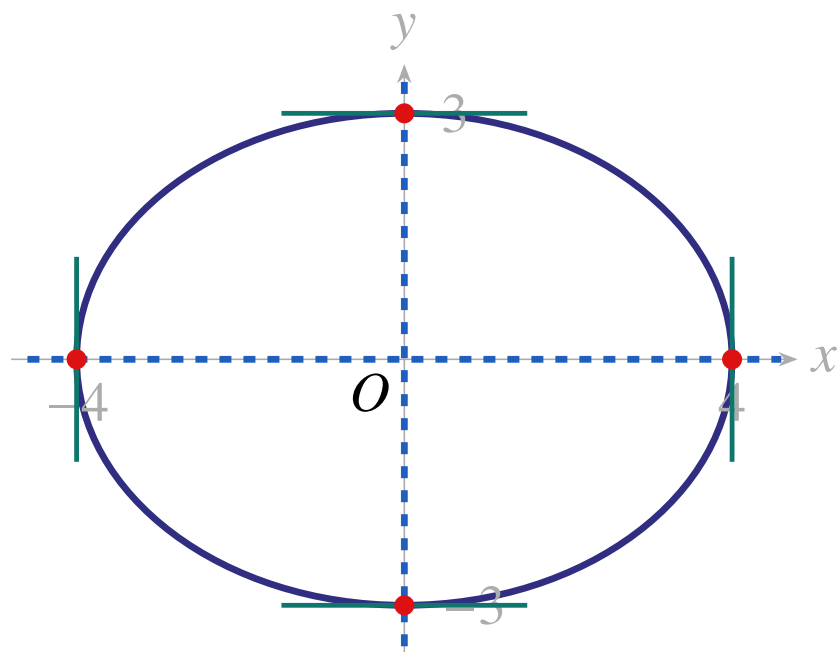
$$\frac{x}{8} + \frac{2y}{9}y' = 0 \quad \Rightarrow \quad y' = -\frac{9x}{16y}$$

$$xy \neq 0: \quad m_N = \frac{16y}{9x}, \quad m_{OP} = \frac{y}{x}$$

$$\frac{16y}{9x} = \frac{y}{x} \quad \Rightarrow \quad 16y = 9y \quad \Rightarrow \quad y = 0$$

Contradiction: we assumed the point was off both axes.

Axis points complete the argument



Ellipse with tangent and normal lines at the four vertices.

$$\frac{dx}{dy} = -\frac{16y}{9x} = 0 \text{ at } (\pm 4, 0)$$

$$T: x = \pm 4, \quad N: y = 0$$

$$y' = 0 \text{ at } (0, \pm 3)$$

$$T: y = \pm 3, \quad N: x = 0$$

$$(\pm 4, 0), \quad (0, \pm 3)$$

What changes for a circle?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$$

When does **every normal** pass through the center?

$$xy \neq 0: \quad m_N = \frac{a^2}{b^2} \frac{y}{x}$$

Equal semiaxes: every point.

Unequal semiaxes: four vertices.

Exit ticket · What does the number prove?

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 3x - 1, & x > 1 \end{cases}$$

A symmetric numerical routine reports approximately **2.5**:

$$D_h = \frac{f(1+h) - f(1-h)}{2h}$$

Does the derivative at input 1 exist?

Is the tangent vertical? **Justify.**

Decide first. Differentiate. Check the line.

The **point** locates the line.

The **derivative** gives the tangent slope.

Perpendicularity determines the normal.

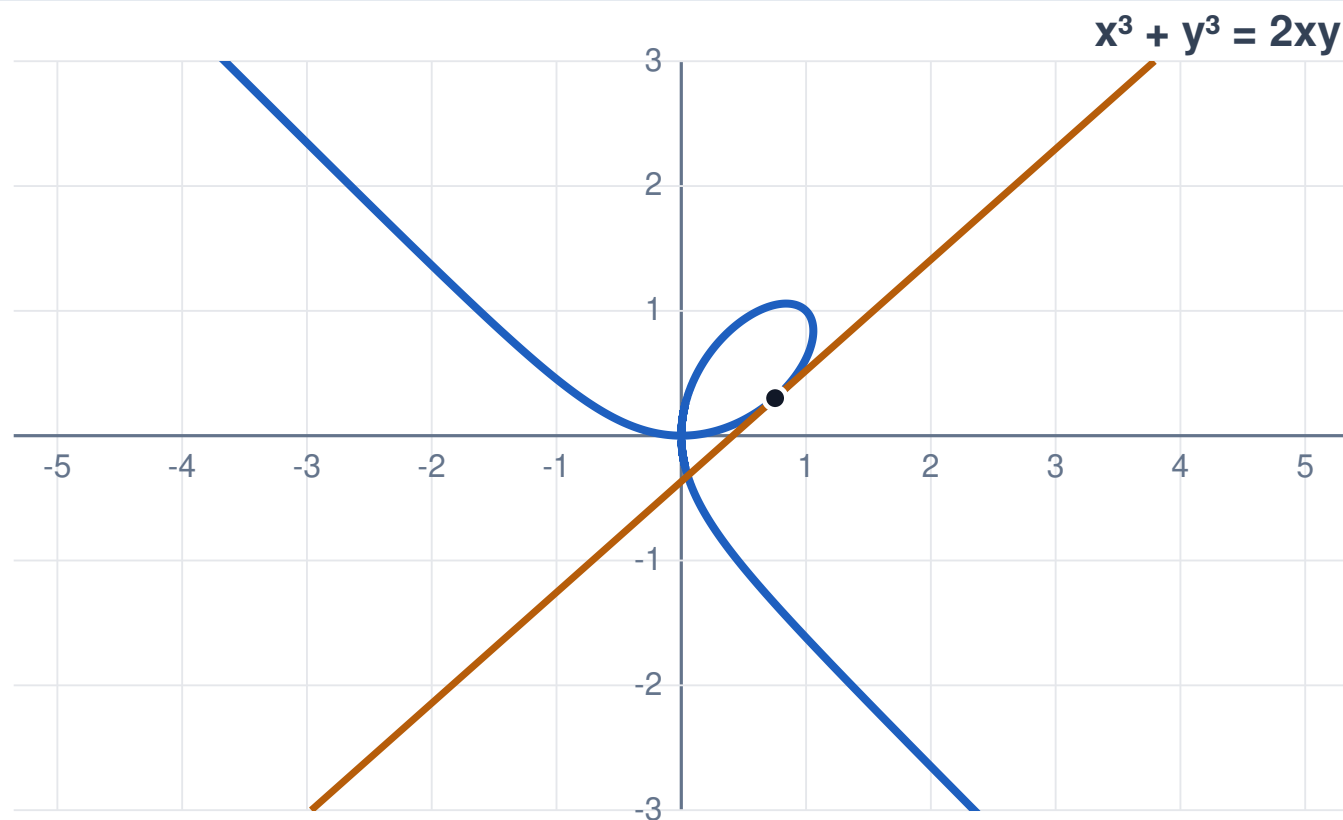
Homework 2.16 — Mixed Differentiation, Tangent and Normal Lines

Questions **1–6, including 6***. No calculator. Show all work for full credit.

Optional explorations: use the Live graphs menu after the lesson.

A moving point, tangent, and normal

Static print state: $t = 0.40$; tangent shown; normal hidden.



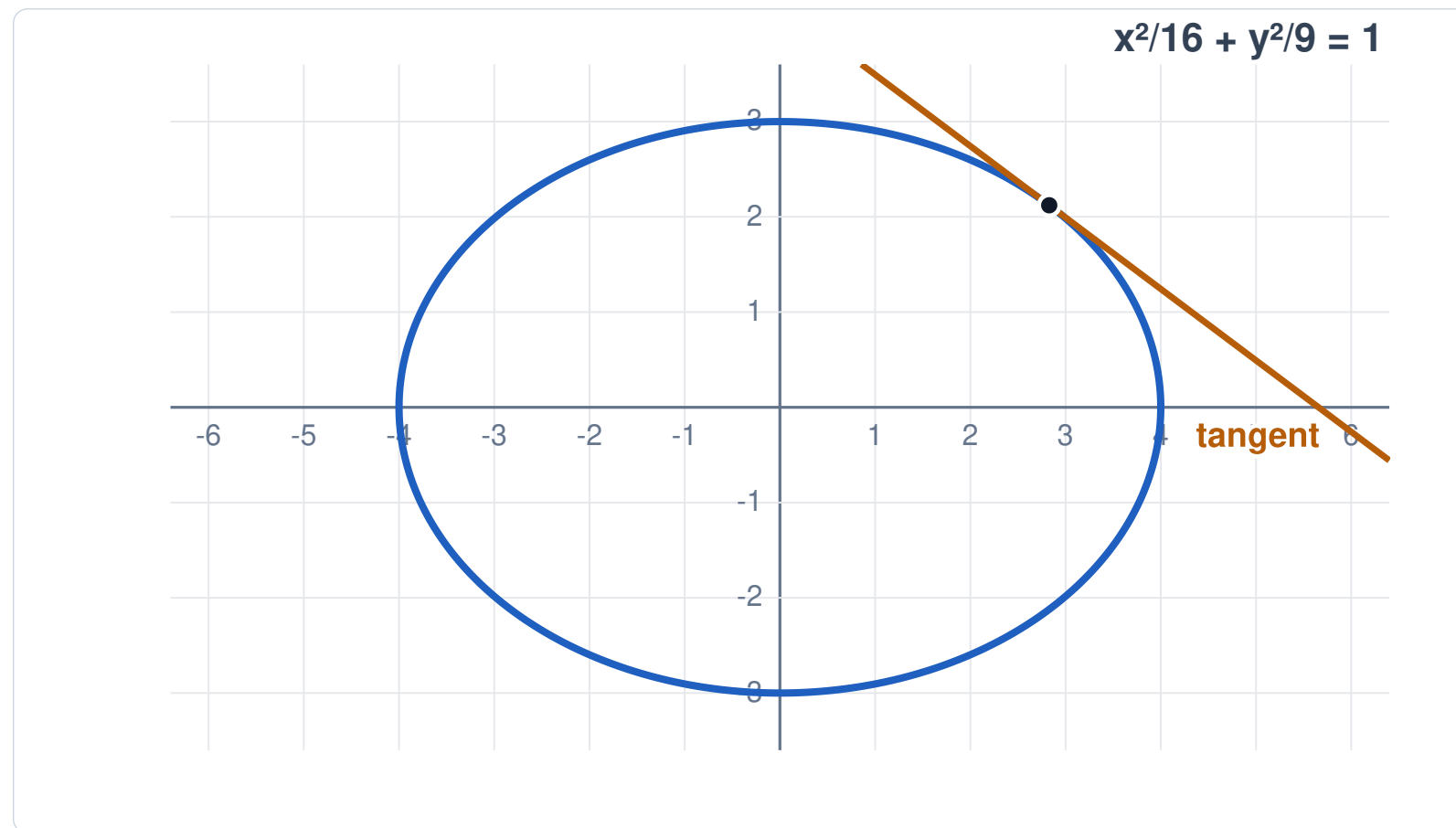
Parameter t

0.4

$P = (0.752, 0.301)$; tangent slope
0.8881; normal slope -1.126.

Test the ellipse claim

Static print state: $\theta = 45^\circ$; tangent shown; normal and radial segment hidden.

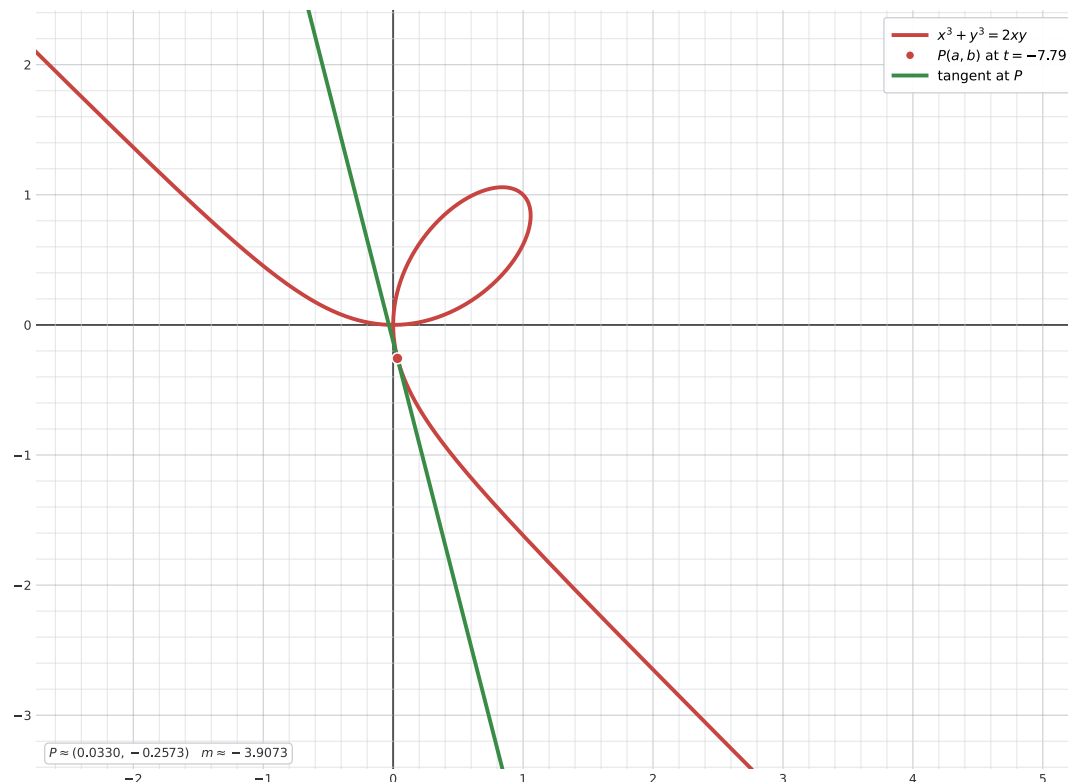


Angle θ

45°

$P = (2.828, 2.121)$; tangent slope
-0.75; normal slope 1.3333.

Your original Desmos graph



Static print state: $t = -7.79$, with the saved point and tangent. [Open the original interactive graph.](#)